

FAMILIES OF IRREDUCIBLE SINGULAR GELFAND-TSETLIN MODULES OF $\mathfrak{gl}(n)$

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ABSTRACT. We prove a conjecture for the irreducibility of singular Gelfand-Tsetlin modules announced in [9]. We describe explicitly the irreducible subquotients of certain classes of singular Gelfand-Tsetlin modules.

1. INTRODUCTION

In 1950, I. Gelfand and M. Tsetlin [11] constructed a basis for any irreducible finite-dimensional $\mathfrak{gl}(n)$ -module. These bases are parameterized by the so-called Gelfand-Tsetlin tableaux, whose entries satisfy some integer relations. Observing that the coefficients in the Gelfand-Tsetlin formulas are rational functions on the entries of the tableaux, it is natural to extend the Gelfand-Tsetlin construction to more general modules. For a generic tableau $T(L)$ (no integer relations between elements of the same row) the corresponding module was constructed in [2] and explicit bases for their irreducible subquotients were given in [7] providing explicitly new irreducible modules for $\mathfrak{gl}(n)$. The first attempt of generalization is the case when there is only one row and a unique pair of entries in this row with integral condition (1-singularity). This case was treated in [9]. In this construction besides regular tableaux appear new tableaux which are called derivatives tableaux. The vector space $V(T(\bar{v}))$ generated by regular and derivatives tableaux has a $\mathfrak{gl}(n)$ -module structure. In [9], V. Futorny, D. Grantcharov and L.E. Ramirez gave sufficient condition for irreducibility of the module $V(T(\bar{v}))$ and conjectured that this same condition is necessary. In particular, for $n = 3$ this was shown in [7].

Conjecture: Let $n \geq 2$, $V(T(\bar{v}))$ the 1-singular $\mathfrak{gl}(n)$ -module. If $V(T(\bar{v}))$ is irreducible then the differences between elements of neighboring rows of $T(\bar{v})$ are not integers.

In the current paper we give a positive answer to this conjecture and describe the irreducible subquotients for certain families of 1-singular modules $V(T(\bar{v}))$.

The paper is organized as follows. In Section 2 we introduce the necessary definitions and notations used through the paper. In Section 3 we recall the definition of Gelfand-Tsetlin subalgebras and Gelfand-Tsetlin modules. In the same section we recall the Gelfand-Tsetlin theorem about realizations of irreducible finite dimensional modules via Gelfand-Tsetlin tableaux. Also, we recall the construction

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of generic Gelfand-Tsetlin modules presented in [2]. In Section 4 we discuss 1-singular Gelfand-Tsetlin tableaux and modules constructed in [9]. In the following section we define a preorder relation in the set of all tableaux (regular end derivatives) and establish important properties of this relation that will be used in the next section. Finally, in Sections 6 and 7 we establish main results in this paper; the first result gives us an explicit basis for an irreducible subquotient of the 1-singular module $V(T(\bar{v}))$ that contains a given tableau, and the second result gives a positive answer for the Conjecture.

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2. CONVENTIONS AND NOTATION

The ground field will be \mathbb{C} . For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers m such that $m \geq a$. We fix an integer $n \geq 2$. By $\mathfrak{gl}(n)$ we denote the general linear Lie algebra consisting of all $n \times n$ complex matrices, and by $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ - the standard basis of $\mathfrak{gl}(n)$ of elementary matrices. We fix the standard triangular decomposition and the corresponding basis of simple roots of $\mathfrak{gl}(n)$. The weights of $\mathfrak{gl}(n)$ will be written as n -tuples $(\lambda_1, \dots, \lambda_n)$.

For a Lie algebra \mathfrak{a} by $U(\mathfrak{a})$ we denote the universal enveloping algebra of \mathfrak{a} . Throughout this paper $U = U(\mathfrak{gl}(n))$. For a commutative ring R , by $\text{Specm } R$ we denote the set of maximal ideals of R .

We will write the vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$ in the following form:

$$v = (v_{n1}, \dots, v_{nn} \mid v_{n-1,1}, \dots, v_{n-1,n-1} \mid \dots \mid v_{21}, v_{22} \mid v_{11}).$$

For $1 \leq j \leq i \leq n$, $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{m\ell}$ are zero. For $i > 0$ by S_i we denote the i th symmetric group. By (m, ℓ) we denote the transposition of S_i switching m and ℓ .

3. GELFAND-TSETLIN MODULES

3.1. Definitions. Recall that $U = U(\mathfrak{gl}(n))$. Let for $m \leq n$, let \mathfrak{gl}_m be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \mid i, j = 1, \dots, m\}$. We have the following chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_n,$$

which induces the chain $U_1 \subset U_2 \subset \dots \subset U_n$ of the universal enveloping algebras $U_m = U(\mathfrak{gl}_m)$, $1 \leq m \leq n$. Let Z_m be the center of U_m . Then Z_m is the polynomial algebra in the m variables $\{c_{mk} \mid k = 1, \dots, m\}$,

$$(1) \quad c_{mk} = \sum_{(i_1, \dots, i_k) \in \{1, \dots, m\}^k} E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_k i_1}.$$

Following [2], we call the subalgebra of U generated by $\{Z_m \mid m = 1, \dots, n\}$ the (*standard*) *Gelfand-Tsetlin subalgebra* of U and will be denoted by Γ . In fact, Γ is the polynomial algebra in the $\frac{n(n+1)}{2}$ variables $\{c_{ij} \mid 1 \leq j \leq i \leq n\}$ ([24]). Let Λ be the polynomial algebra in the variables $\{\lambda_{ij} \mid 1 \leq j \leq i \leq n\}$.

Let $\iota : \Gamma \rightarrow \Lambda$ be the embedding defined by $\iota(c_{mk}) = \gamma_{mk}(\lambda)$, where

$$(2) \quad \gamma_{mk}(\lambda) := \sum_{i=1}^m (\lambda_{mi} + m - 1)^k \prod_{j \neq i} \left(1 - \frac{1}{\lambda_{mi} - \lambda_{mj}} \right).$$

The image of ι coincides with the subalgebra of G -invariant polynomials in Λ , where $G := S_n \times \cdots \times S_1$ ([24]) which we identify with Γ .

Definition 3.1. *A finitely generated U -module M is called a Gelfand-Tsetlin module (with respect to Γ) if M splits into a direct sum of Γ -modules:*

$$M = \bigoplus_{\mathfrak{m} \in \text{Specm } \Gamma} M(\mathfrak{m}),$$

where

$$M(\mathfrak{m}) = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Identifying \mathfrak{m} with the homomorphism $\chi : \Gamma \rightarrow \mathbb{C}$ with $\text{Ker } \chi = \mathfrak{m}$, we will call \mathfrak{m} a *Gelfand-Tsetlin character* of M if $M(\mathfrak{m}) \neq 0$, and $\dim M(\mathfrak{m})$ - the *Gelfand-Tsetlin multiplicity* of \mathfrak{m} . The *Gelfand-Tsetlin support* of a Gelfand-Tsetlin module M is the set of all Gelfand-Tsetlin characters of M .

Remark 3.2. *Note that any irreducible Gelfand-Tsetlin module over $\mathfrak{gl}(n)$ is a weight module with respect to the standard Cartan subalgebra \mathfrak{h} spanned by E_{ii} , $i = 1, \dots, n$. In particular, every highest weight module or, more generally, every module from the category \mathcal{O} is a Gelfand-Tsetlin module.*

3.2. Finite dimensional modules for $\mathfrak{gl}(n)$. In this section we recall a classical result of I. Gelfand and M. Tsetlin which provides an explicit basis for every irreducible finite dimensional $\mathfrak{gl}(n)$ -module.

Definition 3.3. *The following array $T(v)$ of complex numbers $\{v_{ij} \mid 1 \leq j \leq i \leq n\}$*

$$\begin{array}{ccccccc}
 \boxed{v_{n1}} & \boxed{v_{n2}} & & \cdots & & \boxed{v_{n,n-1}} & \boxed{v_{nn}} \\
 & \boxed{v_{n-1,1}} & & \cdots & & \boxed{v_{n-1,n-1}} & \\
 & & & \cdots & & & \\
 & & & & & & \\
 & & & & & \boxed{v_{21}} & \boxed{v_{22}} \\
 & & & & & \boxed{v_{11}} &
 \end{array}$$

is called a Gelfand-Tsetlin tableau.

A Gelfand-Tsetlin tableau $T(v)$ is called *standard* if:

$$v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{> 0}, \quad \text{for all } 1 \leq i \leq k \leq n.$$

Theorem 3.4 ([11]). *Let $L(\lambda)$ be the finite dimensional irreducible module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$. Then there exists a basis of $L(\lambda)$ consisting of all standard tableaux $T(v)$ with fixed top row $v_{n1} = \lambda_1, v_{n2} = \lambda_2 - 1, \dots, v_{nn} =$*

$\lambda_n - n + 1$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the Gelfand-Tsetlin formulas:

$$(3) \quad \begin{aligned} E_{k,k+1}(T(v)) &= - \sum_{i=1}^k \left(\frac{\prod_{j=1}^{k+1} (v_{ki} - v_{k+1,j})}{\prod_{j \neq i}^k (v_{ki} - v_{kj})} \right) T(v + \delta^{ki}), \\ E_{k+1,k}(T(v)) &= \sum_{i=1}^k \left(\frac{\prod_{j=1}^{k-1} (v_{ki} - v_{k-1,j})}{\prod_{j \neq i}^k (v_{ki} - v_{kj})} \right) T(v - \delta^{ki}), \\ E_{kk}(T(v)) &= \left(k - 1 + \sum_{i=1}^k v_{ki} - \sum_{i=1}^{k-1} v_{k-1,i} \right) T(v). \end{aligned}$$

If the new tableau $T(v \pm \delta^{ki})$ is not standard, then the corresponding summand of $E_{k,k+1}(T(v))$ or $E_{k+1,k}(T(v))$ is zero by definition. Furthermore, the action of generators c_{rs} of Γ defined by (1) is given by,

$$(4) \quad c_{rs}(T(v)) = \gamma_{rs}(v)T(v),$$

where γ_{rs} are defined in (2).

3.3. Generic Gelfand-Tsetlin modules. In the case when all denominators are nonintegers, one can use the same formulas and define a new class of infinite dimensional $\mathfrak{gl}(n)$ -modules: *generic* Gelfand-Tsetlin modules (cf. [2], Section 2.3).

Definition 3.5. A Gelfand-Tsetlin tableau $T(v)$ is called *generic* if $v_{rs} - v_{ru} \notin \mathbb{Z}$ for each $1 \leq s < u \leq r \leq n - 1$.

Theorem 3.6 (§2.3 in [2] and Theorem 2 in [21]). *Let $T(v)$ be a generic Gelfand-Tsetlin tableau. Denote by $V(T(v))$ the vector space with basis consisting of all Gelfand-Tsetlin tableaux $T(w)$ satisfying $w_{nj} = v_{nj}$, $w_{ij} - v_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i \leq n - 1$.*

- (i) *The vector space $V(T(v))$ has a structure of a $\mathfrak{gl}(n)$ -module with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand-Tsetlin formulas (3). The module $V(T(v))$ has finite length.*
- (ii) *The action of the generators of Γ on the basis elements of $V(T(v))$ is given by (4).*
- (iii) *The module defined in (i) is a Gelfand-Tsetlin module all of whose Gelfand-Tsetlin multiplicities are 1.*
- (iv) *The action of the generators of Γ is given by (4).*

We will denote the module constructed in Theorem 3.6 by $V(T(v))$. Note that $V(T(v))$ need not to be irreducible. Because Γ has simple spectrum on $V(T(v))$ for $T(w)$ in $V(T(v))$ we may define the *irreducible $\mathfrak{gl}(n)$ -module in $V(T(v))$ containing $T(w)$* to be the subquotient of $V(T(v))$ containing $T(w)$ (see Theorem 3.6(i)). A basis for the irreducible subquotients of $V(T(v))$ can be described in terms of the following set.

$$(5) \quad \Omega^+(T(w)) := \{(r, s, u) \mid w_{rs} - w_{r-1,u} \in \mathbb{Z}_{\geq 0}\}.$$

Theorem 3.7 (Theorem 6.14, in [8]). *Let $T(v)$ be a generic tableau and let $T(w)$ be a tableau in $V(T(v))$. Then the following hold.*

- (i) *The submodule of $V(T(v))$ generated by $T(w)$ has basis*

$$\mathcal{N}(T(w)) := \{T(w') \in V(T(v)) \mid \Omega^+(T(w)) \subseteq \Omega^+(T(w'))\};$$

(ii) The irreducible $\mathfrak{gl}(n)$ -module in $V(T(v))$ containing $T(w)$ has basis

$$\mathcal{I}(T(w)) := \{T(w') \in V(T(v)) \mid \Omega^+(T(w)) = \Omega^+(T(w'))\}.$$

The action of $\mathfrak{gl}(n)$ on both $\mathcal{N}(T(w))$ and $\mathcal{I}(T(w))$ is given by the Gelfand-Tsetlin formulas.

3.4. Gelfand-Tsetlin formulas in terms of permutations. In this subsection we rewrite and generalize the Gelfand-Tsetlin formulas in Theorem 3.4 in convenient for us terms.

Recall the convention that for a vector $v = (v_{n1}, \dots, v_{nn} \mid \dots \mid v_{11})$ in $\mathbb{C}^{\frac{n(n+1)}{2}}$, by $T(v)$, we denote the corresponding to v Gelfand-Tsetlin tableau. Let us call v in $\mathbb{C}^{\frac{n(n+1)}{2}}$ *generic* if $T(v)$ is a generic Gelfand-Tsetlin tableau, and denote by $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}}$ the set of all generic vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$.

Remember that $G = S_n \times \dots \times S_1$. Let \tilde{S}_m denotes the subset of S_m consisting of the transpositions $(1, i)$, $i = 1, \dots, m$. Also, we consider every $\sigma \in \tilde{S}_m$ as an element of G by letting $\sigma[t] = \text{Id}$ whenever $t \neq m$.

Definition 3.8. Let $1 \leq r < n - 1$. Set

$$\varepsilon_{r,r+1} := \delta^{r,1} \in \mathbb{Z}^{\frac{n(n+1)}{2}}.$$

Furthermore, define $\varepsilon_{rr} = 0$ and $\varepsilon_{r+1,r} = -\delta^{r,1}$.

Definition 3.9. For each generic vector w and any $1 \leq r, s \leq n$ we define

$$(6) \quad \begin{aligned} e_{r,r+1}(w) &:= -\frac{\prod_{j=1}^{r+1}(w_{r1} - w_{r+1,j})}{\prod_{j \neq 1}^r (w_{r1} - w_{rj})}, \\ e_{r+1,r}(w) &:= \frac{\prod_{j=1}^{r-1}(w_{r1} - w_{k-1,j})}{\prod_{j \neq 1}^k (w_{r1} - w_{rj})}, \\ e_{rr}(w) &:= r - 1 + \sum_{i=1}^r w_{ri} - \sum_{i=1}^{r-1} w_{r-1,i} \end{aligned}$$

By using permutations we can rewrite the Gelfand-Tsetlin formulas (3) as follows:

Proposition 3.10. Let $v \in \mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}}$. The Gelfand-Tsetlin formulas for the generic Gelfand Tsetlin $\mathfrak{gl}(n)$ -module $V(T(v))$ can be written as follows:

$$E_{\ell m}(T(v+z)) = \sum_{\sigma \in \tilde{S}_r} e_{\ell m}(\sigma(v+z))T(v+z+\sigma(\varepsilon_{\ell m})),$$

where $(\ell, m) \in \{(r, r+1), (r+1, r), (r, r)\}$ and $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ has top row zero.

4. SINGULAR GELFAND-TSETLIN MODULES

In this section we will remember the construction of singular Gelfand-Tsetlin modules given in [9].

Definition 4.1. A vector $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$ will be called *singular* if there exist $1 \leq s < t \leq r \leq n-1$ such that $v_{rs} - v_{rt} \in \mathbb{Z}$. The vector v will be called *1-singular* if there exist i, j, k with $1 \leq i < j \leq k \leq n-1$ such that $v_{ki} - v_{kj} \in \mathbb{Z}$ and $v_{rs} - v_{rt} \notin \mathbb{Z}$ for all $(r, s, t) \neq (k, i, j)$.

From now on we fix (i, j, k) such that $1 \leq i < j \leq k \leq n-1$. In [9], associated with any 1-singular tableau $T(\bar{v})$ is constructed a Gelfand-Tsetlin module $V(T(\bar{v}))$ and explicit formulas for the action of the generators of $\mathfrak{gl}(n)$ and the generators of Γ is given, in this section we will remember that construction.

Let us fix a 1-singular vector \bar{v} such that $\bar{v}_{ki} - \bar{v}_{kj} = 0$. From now on by τ we denote the element $(\tau[n], \dots, \tau[2], \tau[1])$ in G such that $\tau[k]$ is the transposition (i, j) and all other $\tau[t]$ are Id . We formally introduce new tableaux $\mathcal{DT}(\bar{v} + w)$ for every $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ subject to the relations $\mathcal{DT}(\bar{v} + w) + \mathcal{DT}(\bar{v} + \tau(w)) = 0$. We call $\mathcal{DT}(\bar{v} + w)$ the derivative Gelfand-Tsetlin tableau associated with w .

Definition 4.2. We set $V(T(\bar{v}))$ to be the vector space spanned by the set of tableaux $\{T(\bar{v} + w), \mathcal{DT}(\bar{v} + w) \mid w \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}$, subject to the relations $T(\bar{v} + w) = T(\bar{v} + \tau(w))$ and $\mathcal{DT}(\bar{v} + w) + \mathcal{DT}(\bar{v} + \tau(w)) = 0$. We also fix a basis of $V(T(\bar{v}))$ to be the set $\{Tab(w) : w \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}$, where

$$Tab(w) := \begin{cases} T(\bar{v} + w), & \text{if } w_{ki} - w_{kj} \leq 0 \\ \mathcal{DT}(\bar{v} + w), & \text{if } w_{ki} - w_{kj} > 0 \end{cases}$$

For a variables vector v and f is a rational function on variables v_{rs} which is smooth on the hyperplane $v_{ki} - v_{kj} = 0$ we can define the linear map

$$\mathcal{D}^{\bar{v}}(fT(v + z)) = \mathcal{D}^{\bar{v}}(f)T(\bar{v} + z) + f(\bar{v})\mathcal{DT}(\bar{v} + z),$$

where $\mathcal{D}^{\bar{v}}(f) = \frac{1}{2} \left(\frac{\partial f}{\partial v_{ki}} - \frac{\partial f}{\partial v_{kj}} \right) (\bar{v})$. The following lemma will be useful in order to do some computations.

Lemma 4.3. Let f be a rational function on variables v_{rs} smooth on the hyperplane $v_{ki} - v_{kj} = 0$. Then,

- (i) $\mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})f) = f(\bar{v})$.
- (ii) If f is symmetric with respect to v_{ki} and v_{kj} then, $\mathcal{D}^{\bar{v}}(f) = 0$.

Theorem 4.4 ([9] Theorems 4.11 and 4.12). $V(T(\bar{v}))$ is an 1-singular Gelfand-Tsetlin $\mathfrak{gl}(n)$ -module, with action of the generators of $\mathfrak{gl}(n)$ given by

$$(7) \quad E_{rs}(T(\bar{v} + z)) = \mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})E_{rs}(T(v + z)))$$

$$(8) \quad E_{rs}(\mathcal{DT}(\bar{v} + z')) = \mathcal{D}^{\bar{v}}(E_{rs}(T(v + z'))),$$

and the action of the generators of Γ can be written explicitly as follows:

$$(9) \quad c_{rs}(T(\bar{v} + z)) = \gamma_{rs}(\bar{v} + z)T(\bar{v} + z)$$

$$(10) \quad c_{rs}(\mathcal{DT}(\bar{v} + z')) = \gamma_{rs}(\bar{v} + z')\mathcal{DT}(\bar{v} + z') + \mathcal{D}^{\bar{v}}(\gamma_{rs}(v + z'))T(\bar{v} + z')$$

for any $z, z' \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ with $z' \neq \tau(z')$.

For some of the generators of $\mathfrak{gl}(n)$ the action on $V(T(\bar{v}))$ coincide with the classical Gelfand-Tsetlin formulas. The following corollary gives some sufficient conditions in order to have this property.

Corollary 4.5. Let z be any element of $\mathbb{Z}^{\frac{n(n-1)}{2}}$.

- (i) For any ℓ, m such that $k < \min\{\ell, m\}$ or $\max\{\ell, m\} \leq k$ we have

$$E_{\ell m}(T(\bar{v} + z)) = \sum_{\sigma \in \Phi_{\ell m}} e_{\ell m}(\sigma(\bar{v} + z))T(\bar{v} + z + \sigma(\varepsilon_{\ell m})).$$

(ii) *The equality*

$$E_{rs}(\mathcal{DT}(\bar{v} + z)) = \sum_{\sigma \in \Phi_{rs}} e_{rs}(\sigma(\bar{v} + z)) \mathcal{DT}(\bar{v} + z + \sigma(\varepsilon_{rs})),$$

holds whenever r, s satisfy:

$$\begin{cases} k \notin \{r, r+1, \dots, s\}, & \text{if } r < s, \\ k \notin \{s, s+1, \dots, r\}, & \text{if } s < r. \end{cases}$$

Proof. The proof will be based on the basic properties of the rational functions $e_{ij}(w)$ defined in 3.9.

- (i) If $k < \min\{\ell, m\}$ or $\max\{\ell, m\} - 1 < k$ then, $e_{\ell m}(\sigma(v + z))$ is a smooth function, for any $\sigma \in \Phi_{\ell m}$ so, by Lemma 4.3 we have $\mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})e_{\ell m}(\sigma(v + z))) = e_{\ell m}(\sigma(\bar{v} + z))$ and $ev(\bar{v})((v_{ki} - v_{kj})e_{\ell m}(\sigma(v + z))) = 0$. Now, by (7) we have the equality.
- (ii) Under this conditions on r, s the function $e_{rs}(\sigma(v + z))$ is symmetric with respect to v_{ki} and v_{kj} for any $\sigma \in \Phi_{rs}$. Therefore, by Lemma 4.3, $\mathcal{D}^{\bar{v}}e_{rs}(\sigma(v + z)) = 0$ and then, by (8) we have the desired equality.

□

Remark 4.6. Note that $\chi \in \text{Supp}_{GT}(V(T(\bar{v})))$ if and only if there exist $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ such that $\chi(c_{rs}) = \gamma_{rs}(\bar{v} + w)$ for any $1 \leq s \leq r \leq n$. In this situation each Gelfand-Tsetlin subspace $V(T(\bar{v}))(\chi)$ is generated by $T(\bar{v} + w)$ and $\mathcal{DT}(\bar{v} + w)$.

Remark 4.7. As the polynomials $\{\gamma_{rs}(v)\}_{1 \leq s \leq r \leq n}$ are symmetric in the entries of v and generated all symmetric polynomials, we have $\gamma_{rs}(v) = \gamma_{rs}(v')$ for any r, s if only if, $v = \sigma(v')$ for some $\sigma \in G$. In particular, for the 1-singular vector \bar{v} we have $\gamma_{rs}(\bar{v} + z) = \gamma_{rs}(\bar{v} + w)$ for any $1 \leq s \leq r \leq n$ if, and only if, $w = z$ or $w = \tau(z)$.

Lemma 4.8 ([9] Lemma 5.2). Assume $\tau(z) \neq z$. Then:

- (i) $(c_{k2} - \gamma_{k2}(\bar{v} + z))T(\bar{v} + z) = 0$.
- (ii) $(c_{k2} - \gamma_{k2}(\bar{v} + z))\mathcal{DT}(\bar{v} + z) = \mathcal{D}^{\bar{v}}(\gamma_{k2}(v + z))T(\bar{v} + z)$ with $\mathcal{D}^{\bar{v}}(\gamma_{k2}(v + z)) \neq 0$.
- (iii) $(c_{k2} - \gamma_{k2}(\bar{v} + z))^2\mathcal{DT}(\bar{v} + z) = 0$.

5. Γ SEPARATES TABLEAUX IN $V(T(\bar{v}))$

One essential property of generic Gelfand-Tsetlin modules described in Theorem 3.6 is that for any two different tableaux in $V(T(\bar{v}))$ there exists an element γ of Γ that separates those tableaux (i.e. the action of γ has different eigenvalues on this two tableaux). In this section we will give a detailed prove of this fact for any 1-singular module $V(T(\bar{v}))$. Remember that we fix a basis $\{Tab(z) \mid z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}$ for $V(T(\bar{v}))$.

For any $1 \leq s \leq r \leq n$ and $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ we will denote by $C_{rs}(z)$ the element $c_{rs} - \gamma_{rs}(\bar{v} + z)$ of Γ .

Lemma 5.1. For any $1 \leq s \leq r \leq n$, any $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ and $\chi \in \text{Supp}_{GT}(V(T(\bar{v})))$, the subspace $V(T(\bar{v}))(\chi)$ is $C_{rs}(z)$ -invariant.

Proof. By Remark 4.6 any Gelfand-Tsetlin subspace $V(T(\bar{v}))(\chi)$ is generated by $T(\bar{v} + w)$ and $\mathcal{DT}(\bar{v} + w)$ for some $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$. Now,

$$\begin{aligned} C_{rs}(z)T(\bar{v} + w) &= (\gamma_{rs}(\bar{v} + w) - \gamma_{rs}(\bar{v} + z))T(\bar{v} + w) \\ C_{rs}(z)\mathcal{DT}(\bar{v} + w) &= \mathcal{D}^{\bar{v}}(\gamma_{rs}(\bar{v} + z))T(\bar{v} + w) + (\gamma_{rs}(\bar{v} + w) - \gamma_{rs}(\bar{v} + z))\mathcal{DT}(\bar{v} + w). \end{aligned}$$

□

Definition 5.2. Given $z, w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, we write $Tab(z) \prec Tab(w)$ if, and only if, there exists $u \in U(\mathfrak{gl}(n))$ such that $Tab(w)$ appears with nonzero coefficient in the decomposition of $u \cdot Tab(z)$ as a linear combination of tableaux.

Lemma 5.3. Let $w \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ be such that $w \neq \tau(w)$, then $\mathcal{DT}(\bar{v} + w) \prec T(\bar{v} + w)$.

Proof. By Lemma 4.8(ii) we have, $C_{k2}(w)\mathcal{DT}(\bar{v} + w) = \mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w))T(\bar{v} + w)$. As $w \neq \tau(w)$, $\mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w)) \neq 0$ and then, $\mathcal{DT}(\bar{v} + w) \prec T(\bar{v} + w)$. □

Lemma 5.4. Let $z, w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ be such that $w \neq z$ and $w \neq \tau(z)$. There exists $\gamma_z^w \in \Gamma$ such that

$$\gamma_z^w \cdot T(\bar{v} + z) = \gamma_z^w \cdot \mathcal{DT}(\bar{v} + z) = 0 \text{ and } \gamma_z^w \cdot Tab(w) = Tab(w)$$

Proof. Let us fix r and s such that $\gamma_{rs}(\bar{v} + w) \neq \gamma_{rs}(\bar{v} + z)$ (such r, s exist because of Remark 4.7). Set $a := \gamma_{rs}(\bar{v} + w) - \gamma_{rs}(\bar{v} + z) \neq 0$, by a direct computation we have the following identities:

- (i) $C_{rs}(z)T(\bar{v} + z) = 0 = C_{rs}^2(z)\mathcal{DT}(\bar{v} + z)$.
- (ii) $C_{rs}(z)\mathcal{DT}(\bar{v} + z) = \mathcal{D}^{\bar{v}}(\gamma_{rs}(v + z))T(\bar{v} + z)$.
- (iii) $C_{rs}(z)T(\bar{v} + w) = aT(\bar{v} + w)$
- (iv) $C_{rs}(z)\mathcal{DT}(\bar{v} + w) = \mathcal{D}^{\bar{v}}(\gamma_{rs}(v + w))T(\bar{v} + w) + a\mathcal{DT}(\bar{v} + w)$.
- (v) $C_{rs}^2(z)T(\bar{v} + w) = a^2T(\bar{v} + w)$
- (vi) $C_{rs}^2(z)\mathcal{DT}(\bar{v} + w) = 2a\mathcal{D}^{\bar{v}}(\gamma_{rs}(v + w))T(\bar{v} + w) + a^2\mathcal{DT}(\bar{v} + w)$.

Now, we have two cases:

Case 1. Suppose $Tab(w) = T(\bar{v} + w)$. In this case we take $\gamma_z^w = \frac{1}{a^2}C_{rs}^2(z)$.

Case 2. Suppose that $Tab(w) = \mathcal{DT}(\bar{v} + w)$. In this case we have two possibilities, namely:

- (i) $\mathcal{D}^{\bar{v}}(\gamma_{rs}(v + w)) = 0$. In this case, from (v) we have $C_{rs}^2(z) \cdot \mathcal{DT}(\bar{v} + w) = a^2\mathcal{DT}(\bar{v} + w)$. So, we can take $\gamma_z^w = \frac{1}{a^2}C_{rs}^2(z)$.
- (ii) $\mathcal{D}^{\bar{v}}(\gamma_{rs}(v + w)) \neq 0$. By Lemma 4.8(ii), $\mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w)) \neq 0$ and

$$\begin{cases} C_{k2}(w)T(\bar{v} + w) &= 0, \\ C_{k2}(w)\mathcal{DT}(\bar{v} + w) &= \mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w))T(\bar{v} + w). \end{cases}$$

So, applying $C_{k2}(w)$ to the equality (vi), we have:

$$(11) \quad C_{k2}(w)C_{rs}^2(z)\mathcal{DT}(\bar{v} + w) = a^2\mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w))T(\bar{v} + w)$$

Now, replacing Equality (11) in (vi), we have:

$$(12) \quad \left(1 - \frac{2C_{k2}(w)}{a\mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w))}\right) C_{rs}^2(z)\mathcal{DT}(\bar{v} + w) = a^2\mathcal{DT}(\bar{v} + w).$$

Therefore, in this case we can consider $\gamma_z^w = \frac{1}{a^2} \left(1 - \frac{2C_{k2}(w)}{a\mathcal{D}^{\bar{v}}(\gamma_{k2}(v + w))}\right) C_{rs}^2(z)$.

Summarizing, we have:

$$\gamma_z^w = \begin{cases} \frac{1}{a^2} C_{rs}^2(z), & \text{if } Tab(w) = T(\bar{v} + w) \text{ or } \\ & \mathcal{D}^{\bar{v}}(\gamma_{rs}(v + w)) = 0 \\ \frac{1}{a^2} \left(1 - \frac{2C_{k2}(w)}{a\mathcal{D}^{\bar{v}}(\gamma_{k2}(v+w))}\right) C_{rs}^2(z), & \text{if } \mathcal{D}^{\bar{v}}(\gamma_{rs}(v + w)) \neq 0 \end{cases}$$

□

Remark 5.5. Note that each γ_z^w on the previous lemma is a combination of products of elements of Γ of the form $C_{\ell m}(w')$. So, by Lemma 5.1, the Gelfand-Tsetlin subspaces $V(T(\bar{v}))(\chi)$ are γ_z^w -invariant.

Lemma 5.6. Let us consider $T := aT(\bar{v} + w) + b\mathcal{D}T(\bar{v} + \tau(w))$, where $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ is such that $w \neq \tau(w)$ and $a, b \in \mathbb{C}$. Then,

- (i) If $a \neq 0$, then $\gamma_1 \cdot T = T(\bar{v} + w)$ for some $\gamma_1 \in \Gamma$.
- (ii) If $b \neq 0$, then $\gamma_2 \cdot T = \mathcal{D}T(\bar{v} + \tau(w))$ for some $\gamma_2 \in \Gamma$.

Proof. Let $\gamma_1, \gamma_2 \in \Gamma$ defined by

$$\gamma_1 = \begin{cases} \frac{1}{a}, & \text{if } b = 0 \\ \frac{C_{rs}^2(w)}{b\mathcal{D}^{\bar{v}}(\gamma_{k2}(v+w))}, & \text{if } b \neq 0 \end{cases}, \quad \gamma_2 = \begin{cases} \frac{1}{b}, & \text{if } a = 0 \\ \frac{1}{b} \left(1 - a \frac{C_{rs}^2(w)}{a\mathcal{D}^{\bar{v}}(\gamma_{k2}(v+w))}\right), & \text{if } a \neq 0 \end{cases}.$$

First we note that by Lemma 4.8, the denominators of γ_1, γ_2 are not zero. The rest of the proof is a straightforward verification □

Theorem 5.7. If $z, w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ are such that $Tab(z) \prec Tab(w)$ then, there exist $u \in U(\mathfrak{gl}(n))$ such that $u \cdot Tab(z) = Tab(w)$.

Proof. As $Tab(z) \prec Tab(w)$, there exist $u' \in U(\mathfrak{gl}(n))$ such that $u' \cdot Tab(z)$ can be written as follows

$$(13) \quad \sum_{i=0}^s a_i T(\bar{v} + w_i) + b_i \mathcal{D}T(\bar{v} + \tau(w_i)) \in V(T(\bar{v}))(\chi_0) \oplus \cdots \oplus V(T(\bar{v}))(\chi_s)$$

where $w_i \neq w_j, \tau(w_j)$ for any $i \neq j$ and $w_0 = w$ or $w_0 = \tau(w)$, $a_0 \neq 0$ or $b_0 \neq 0$, and χ_i the Gelfand-Tsetlin character associated with w_i . By Lemma 5.4, for each $j \in \{1, 2, \dots, s\}$, there exist $\gamma_{w_j}^{w_0} \in \Gamma$ such that

$$\gamma_{w_j}^{w_0} T(\bar{v} + w_j) = \gamma_{w_j}^{w_0} \mathcal{D}T(\bar{v} + w_j) = 0 \text{ and } \gamma_{w_j}^{w_0} Tab(w_0) = Tab(w_0)$$

Then, by Remark 5.5, applying $\gamma := \gamma_{w_s}^{w_0} \cdots \gamma_{w_1}^{w_0}$ to $u' \cdot Tab(z)$, we have

$$\gamma u' \cdot Tab(z) = \gamma_{w_s}^{w_0} \cdots \gamma_{w_1}^{w_0} \left(\sum_{i=1}^s a_i T(\bar{v} + w_i) + b_i \mathcal{D}T(\bar{v} + \tau(w_i)) \right) \in V(T(\bar{v}))(\chi_0)$$

So, $\gamma u' \cdot Tab(z) = aT(\bar{v} + w) + b\mathcal{D}T(\bar{v} + \tau(w))$ for some $a, b \in \mathbb{C}$. Let us see the relation between the coefficients a, b and a_0, b_0 .

Case 1. Suppose $Tab(w) = T(\bar{v} + w)$. In this case, for any $j = 1, \dots, s$, we have $\gamma_{w_j}^w T(\bar{v} + w) = T(\bar{v} + w)$ (by construction of $\gamma_{w_j}^{w_0}$) and by Remark 5.5, $\gamma_{w_j}^w \mathcal{D}T(\bar{v} + w) = \mathcal{D}T(\bar{v} + w)$. Therefore,

$$\gamma u \cdot Tab(z) = a_0 T(\bar{v} + w) + b_0 \mathcal{D}T(\bar{v} + \tau(w)).$$

Now, as $a_0 \neq 0$, by Lemma 5.6 there exist $\gamma_1 \in \Gamma$ such that

$$\gamma_1 \gamma u \cdot Tab(z) = \gamma_1 (a_0 T(\bar{v} + w) + b_0 \mathcal{D}T(\bar{v} + \tau(w))) = T(\bar{v} + w).$$

Case 2. Suppose $Tab(w) = \mathcal{DT}(\bar{v} + w)$. In this case, for any $j = 1, \dots, s$, we have $\gamma_{w_j}^w \mathcal{DT}(\bar{v} + w) = \mathcal{DT}(\bar{v} + w)$ (by construction of $\gamma_{w_j}^{w_0}$) and $\gamma_{w_j}^w T(\bar{v} + w) = \alpha_j T(\bar{v} + w)$ for some $\alpha_j \in \mathbb{C}$. Therefore,

$$\gamma u \cdot Tab(z) = aT(\bar{v} + w) + b_0 \mathcal{DT}(\bar{v} + \tau(w)),$$

with $a = \alpha_s \cdots \alpha_1 a_0$ and $b_0 \neq 0$. By Lemma 5.6 there exist $\gamma_2 \in \Gamma$ such that $\gamma_2 \gamma u \cdot Tab(z) = \gamma_2(aT(\bar{v} + w) + b_0 \mathcal{DT}(\bar{v} + \tau(w))) = \mathcal{DT}(\bar{v} + w)$. \square

Corollary 5.8. *The relation “ \prec ” define a preorder on the set of tableaux $\mathcal{B}(T(\bar{v}))$ (i.e. \prec is a reflexive and transitive relation).*

Proof. Reflexivity is clear from the definition of “ \prec ”. For transitivity, assume that $Tab(w_1) \prec Tab(w_2)$ and $Tab(w_2) \prec Tab(w_3)$ for some $w_1, w_2, w_3 \in \mathbb{Z}^{\frac{n(n-1)}{2}}$. By Theorem 5.7 there exists $u_1, u_2 \in U(\mathfrak{gl}(n))$ such that $u_1 Tab(w_1) = Tab(w_2)$ and $u_2 Tab(w_2) = Tab(w_3)$. Therefore, $u_2 u_1 Tab(w_1) = Tab(w_3)$. That is, $Tab(w_1) \prec Tab(w_3)$ \square

6. IRREDUCIBLE SUBQUOTIENTS IN $V(T(\bar{v}))$

The Theorem 3.7 provides an explicit basis for an irreducible submodule that contains a given tableau for generic case. In this section we will present a similar result for 1-singular case and this will lead us an alternative proof for Theorem 4.14 in [9].

Definition 6.1. *Given $z, w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, define the **distance** between the tableaux, $Tab(z)$ and $Tab(w)$ by*

$$d(z, w) = \sum_{1 \leq s \leq r \leq n} |z_{rs} - w_{rs}|.$$

The Theorem 4.14 in [9] states that if $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$, for every (r, s, t) , then the module $V(T(\bar{v}))$ is irreducible. Now consider a tableau such that the condition $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$, for every (r, s, t) is satisfied for any $r \geq k+1$. We will show as construct a basis for a irreducible subquotient of $V(T(\bar{v}))$ that contains a given tableaux. For this we need of some definitions.

Definition 6.2. *For $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, we define*

$$\begin{aligned} \Omega_k(Tab(w)) &= \{(r, s, t) \mid r \leq k, (\bar{v}_{rs} + w_{rs}) - (\bar{v}_{r-1,t} + w_{r-1,t}) \in \mathbb{Z}\} \\ \Omega_k^+(Tab(w)) &= \{(r, s, t) \mid r \leq k, (\bar{v}_{rs} + w_{rs}) - (\bar{v}_{r-1,t} + w_{r-1,t}) \in \mathbb{Z}_{\geq 0}\} \\ \mathcal{I}_k(Tab(w)) &= \{Tab(w') \in V(T(\bar{v})) \mid \Omega_k^+(Tab(w)) = \Omega_k^+(Tab(w'))\} \end{aligned}$$

Lemma 6.3. *Let $Tab(w'), Tab(w^*) \in \mathcal{I}_k(Tab(w))$ be tableaux such that $Tab(w') \not\prec Tab(w^*)$, then there exists i, j with $k+1 \leq j \leq i < n$ such that $w'_{ij} \neq w^*_{ij}$.*

Proof. The tableaux $Tab(w'), Tab(w^*) \in \mathcal{I}_k(Tab(w))$ can be separated in two parts: the top part, i.e. the part from $(k+1)$ -th row to n -th row and the bottom part from row 1 to row k . Now, suppose that $w'_{ij} = w^*_{ij}$ for any $k+1 \leq j \leq i < n$. In this case we have $Tab(w') \prec Tab(w^*)$, because $Tab(w'), Tab(w^*) \in \mathcal{I}_k(Tab(w))$ implies (as in the generic case for $\mathfrak{gl}(k)$) that bottom parts of $Tab(w')$ and $Tab(w^*)$ (that we denote by $Tab_k(w'), Tab_k(w^*)$) are such that $Tab_k(w') \prec Tab_k(w^*)$. \square

Theorem 6.4. *The set $\mathcal{I}_k(Tab(w))$ is a basis for irreducible subquotient of $V(T(\bar{v}))$. that contains the tableau $Tab(w)$.*

Proof. Let $N = \text{Span}_{\mathbb{C}} \mathcal{I}_k(\text{Tab}(w))$ be the submodule of $V(T(\bar{v}))$ generated by the set $\mathcal{I}_k(\text{Tab}(w))$. If N is not irreducible. By Corollary 5.8, there exist tableaux $\text{Tab}(z')$ and $\text{Tab}(w)$ such that $\text{Tab}(z') \not\prec \text{Tab}(w)$. Now, fix w and choose $z \in \{z' \in \mathbb{Z}^{\frac{n(n+1)}{2}} \mid \text{Tab}(z') \not\prec \text{Tab}(w)\}$ such that $d(z, w)$ is minimal. Set $d := d(z, w)$ then, if $z' \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is such that $d(z', w) < d$, we should have $\text{Tab}(z') \prec \text{Tab}(w)$. As $\text{Tab}(z) \not\prec \text{Tab}(w)$, we have $d \geq 1$, that is, $z_{rs} \neq w_{rs}$ for some r, s and by Theorem 6.3 we have that is possible for $r \geq k+1$. Now fix the position r, s and assume without loss of generality that $z_{rs} < w_{rs}$. The case $w_{rs} < z_{rs}$ will be analogous.

As $z_{rs} < w_{rs}$, we have $d(z + \delta^{rs}, w) = d - 1 < d$. Then, $\text{Tab}(z + \delta^{rs}) \prec \text{Tab}(w)$. Therefore, as by Corollary 5.8, \prec is transitive, we should have $\text{Tab}(z) \not\prec \text{Tab}(z + \delta^{rs})$. So, if $\text{Tab}(z + \delta^{rs})$ appear in the decomposition of $u \cdot \text{Tab}(z)$ for some $u \in U$, the corresponding coefficient should be zero.

As the formulas that define the action of $\mathfrak{gl}(n)$ on $V(T(\bar{v}))$ depend of the type of tableau (derivative tableau or regular tableau), we will consider four cases as follows

	$\text{Tab}(z)$	$\text{Tab}(z + \delta^{rs})$
Case 1	$T(\bar{v} + z)$	$T(\bar{v} + z + \delta^{rs})$
Case 2	$\mathcal{DT}(\bar{v} + z)$	$\mathcal{DT}(\bar{v} + z + \delta^{rs})$
Case 3	$T(\bar{v} + z)$	$\mathcal{DT}(\bar{v} + z + \delta^{rs})$
Case 4	$\mathcal{DT}(\bar{v} + z)$	$T(\bar{v} + z + \delta^{rs})$

Case 1. The coefficient of $T(\bar{v} + z + \delta^{rs})$ in decomposition $E_{r,r+1}T(\bar{v} + z)$ is given by $\mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})e_{r,r+1}(\sigma_s(v + z)))$ where σ_s is the transposition $(1, s)$ on row r and identity on the other rows. But, $\mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})e_{r,r+1}(\sigma_s(v + z))) = e_{r,r+1}(\sigma_s(\bar{v} + z))$, because in this case the function $e_{r,r+1}(\sigma_s(v + z))$ is smooth. As the numerator of $e_{r,r+1}(\sigma_s(\bar{v} + z))$ is a product of differences of type $(\bar{v}_{rs} + z_{rs}) - (\bar{v}_{r+1,t} + z_{r+1,t})$. We necessarily have $(\bar{v}_{rs} + z_{rs}) - (\bar{v}_{r+1,t} + z_{r+1,t}) = 0$ for some t , that is: $\bar{v}_{r+1,t} - \bar{v}_{r,s} \in \mathbb{Z}$, with $r \geq k+1$, that is a contradiction.

Case 2. The coefficient of $\mathcal{DT}(\bar{v} + z + \delta^{rs})$ in decomposition $E_{r,r+1}\mathcal{DT}(\bar{v} + z)$ is given by $e_{r,r+1}(\sigma_s(\bar{v} + z))$. Analogously the first case, we obtain that $\bar{v}_{r+1,t} - \bar{v}_{r,s} \in \mathbb{Z}$ for some t , with $r \geq k+1$, that is a contradiction.

Case 3. The only possibility in order to have $\text{Tab}(z) = T(\bar{v} + z)$ and $\text{Tab}(z + \delta^{rs}) = \mathcal{DT}(\bar{v} + z + \delta^{rs})$ is $z_{ki} = z_{kj}$ and $(r, s) \in \{(k, i), (k, j)\}$. As the coefficient of $\mathcal{DT}(\bar{v} + z + \delta^{rs})$ is $ev(\bar{v})((v_{ki} - v_{kj})e_{k,k+1}(\sigma_s(v + z)))$ and $e_{k,k+1}(\sigma_s(v + z))$ has singularity at $v_{ki} - v_{kj} = 0$, $ev(\bar{v})((v_{ki} - v_{kj})e_{k,k+1}(\sigma_s(v + z))) = 0$ if, and only if

$$\prod_{j=1}^{k+1} ((\bar{v} + z)_{ks} - (\bar{v} + z)_{k+1,j}) = 0$$

which implies that some difference $\bar{v}_{ks} - \bar{v}_{k+1,t} \in \mathbb{Z}$, with $r \geq k+1$, that is a contradiction.

Case 4. In this case we have:

$$\begin{aligned} E_{r,r+1}(\mathcal{DT}(\bar{v} + z)) &= \sum_{\sigma \in \Phi_{r,r+1}} \mathcal{D}^{\bar{v}}(e_{r,r+1}(\sigma(v + z))T(\bar{v} + z + \sigma(\varepsilon_{r,r+1}))) \\ &+ \sum_{\sigma \in \Phi_{r,r+1}} e_{r,r+1}(\sigma(\bar{v} + z))\mathcal{DT}(\bar{v} + z + \sigma(\varepsilon_{r,r+1})) \end{aligned}$$

- (i) If $e_{r,r+1}(\sigma_s(\bar{v} + z)) = 0$ then, as in Case 1. that implies $\bar{v}_{r+1,t} - \bar{v}_{r,s} \in \mathbb{Z}$ for some t , with $r \geq k+1$, that is a contradiction.
- (ii) If $e_{r,r+1}(\sigma_s(\bar{v} + z)) \neq 0$ then, $\mathcal{DT}(\bar{v} + z) \prec \mathcal{DT}(\bar{v} + z + \delta^{rs})$ and as $c_{k2}(\mathcal{DT}(\bar{v} + z + \delta^{rs})) \prec T(\bar{v} + z + \delta^{rs})$, the coefficient of $T(\bar{v} + z + \delta^{rs})$ on the decomposition of $c_{k2}E_{r,r+1}\mathcal{DT}(\bar{v} + z)$ is not zero. Therefore, $\mathcal{DT}(\bar{v} + z) \prec T(\bar{v} + z + \delta^{rs})$, which contradicts the hypothesis.

Therefore, we have $\bar{v}_{rs} - \bar{v}_{r-1,t} \in \mathbb{Z}$ for some (r, s, t) is satisfied, in this case, with $r \geq k+1$, what is a contradiction. Thus the module N is irreducible. \square

Remark 6.5. Note that if $\Omega_k(\text{Tab}(w)) = \emptyset$, Theorem 6.4 gives an alternative proof of Theorem 4.14 in [9] which gives a sufficient condition on the entries of \bar{v} in order to have the irreducibility of $V(T(\bar{v}))$.

7. IRREDUCIBILITY OF $V(T(\bar{v}))$

Now we will prove that the conditions given in Theorem 4.14 in [9] are necessary for irreducibility of the module $V(T(\bar{v}))$. For this we need some definitions and lemmas.

Definition 7.1. For any $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ we write

$$\Omega^+(Tab(w)) := \{(r, s, t) \mid (\bar{v}_{rs} + w_{rs}) - (\bar{v}_{r-1,t} + w_{r-1,t}) \in \mathbb{Z}_{\geq 0}\}$$

Remark 7.2. Note that, in the case of generic modules, if $Tab(z) \prec Tab(w)$ implies $|\Omega^+(Tab(z))| \leq |\Omega^+(Tab(w))|$ (see Theorem 3.7(i)). However, for singular modules we can have $|\Omega^+(Tab(z))| - 1 = |\Omega^+(Tab(w))|$. In fact, consider $\bar{v} = (a, b, c, x, x, x)$ such that $\{a - x, b - x, c - x\} \cap \mathbb{Z} = \emptyset$ and $w = (0, 0, 0)$, then $|\Omega^+(Tab(z))| = 2$ while $E_{32}Tab(z) = Tab(z - \delta^{21})$ and $|\Omega^+(Tab(z - \delta^{21}))| = 1$.

The following lemma shows that $|\Omega^+(Tab(z))| - 1$ is the infimum for the size of $\Omega^+(Tab(w))$ for any $Tab(w) \in U \cdot Tab(z)$

Proposition 7.3. Let $Tab(z)$ and $Tab(w)$ be tableaux in $\mathcal{B}(T(\bar{v}))$ such that $Tab(w) \in U \cdot Tab(z)$, then $|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1$.

To prove this proposition we will prove first that the lemma is true when $Tab(z) \prec_g Tab(w)$ for some $g \in \mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$.

Lemma 7.4. Let $Tab(z)$ and $Tab(w)$ be tableaux in $\mathcal{B}(T(\bar{v}))$ such that $Tab(z) \prec_g Tab(w)$ for some $g \in \mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ with $1 \leq r \leq n-1$, then $|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1$.

Proof. We will analyze the action of generators of $\mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ in all tableaux $Tab(z) \in V(T(\bar{v}))$ such that $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(z))| - 1$ and $Tab(z) \prec Tab(w)$. A case by case verification we have that, the coefficient of tableau $Tab(w)$ is equal to zero whenever $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(z))| - 2$ and the list of all possible tableaux $Tab(z)$ and $Tab(w)$ such that $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$ and the coefficient of $Tab(w)$ is not zero is the following:

- (I) $\mathcal{D} \begin{pmatrix} x & & x+a \\ & x & \\ & & x+1 \end{pmatrix} \prec_{E_{k-1,k}} \begin{pmatrix} x & & x+a \\ & x+1 & \\ & & x+1 \end{pmatrix}, \quad a \in \mathbb{Z}_{<0}.$
- (II) $\mathcal{D} \begin{pmatrix} & x & \\ x & & x+a \\ & x+a & \end{pmatrix} \prec_{E_{k-1,k}} \begin{pmatrix} & x & \\ x+a & & x+1 \\ & x+1 & \end{pmatrix}, \quad a \in \mathbb{Z}_{<0}.$

$$(III) \quad \mathcal{D} \begin{pmatrix} x & x+a \\ & x \end{pmatrix} \prec_{E_{k+1,k}} \begin{pmatrix} x-1 & x+a \\ & x \end{pmatrix}, \quad a \in \mathbb{Z}_{<0}.$$

$$(IV) \quad \begin{pmatrix} x & \\ x & x \end{pmatrix} \prec_{E_{k-1,k}} \begin{pmatrix} x & \\ x & x+1 \end{pmatrix}.$$

$$(V) \quad \begin{pmatrix} x & x \\ & x \end{pmatrix} \prec_{E_{k+1,k}} \begin{pmatrix} x-1 & x \\ & x \end{pmatrix}.$$

Where configurations above represent the part of the tableaux around row k . First of all, by Corollary 4.5 is enough to consider $r \in \{k, k-1\}$, in fact, for the other cases the action is given by the classical Gelfand-Tsetlin formulas, as in the generic case. So we have $|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))|$ (see Theorem 3.7(i)).

Assume first that $g = E_{r,r+1}$. The action of $E_{r,r+1}$ on basis elements of $V(T(\bar{v}))$ is given by.

$$(14) \quad \begin{aligned} E_{r,r+1}T(\bar{v} + w) &= \sum_{\sigma} \mathcal{D}^{\bar{v}}((v_{ri} - v_{rj})e_{r,r+1}(\sigma(v+w))T(\bar{v} + w + \sigma(\delta^{r1})) + \\ &\quad \sum_{\sigma} ((v_{ri} - v_{rj})e_{r,r+1}(\sigma(v+w))) (\bar{v}) \mathcal{DT}(\bar{v} + w + \sigma(\delta^{r1})), \end{aligned}$$

$$(15) \quad \begin{aligned} E_{r,r+1}\mathcal{DT}(\bar{v} + w) &= \sum_{\sigma} \mathcal{D}^{\bar{v}}(e_{r,r+1}(\sigma(v+w))T(\bar{v} + w + \sigma(\delta^{r1})) \\ &\quad + \sum_{\sigma} e_{r,r+1}(\sigma(\bar{v} + w))\mathcal{DT}(\bar{v} + w + \sigma(\delta^{r1})). \end{aligned}$$

Depending on the $Tab(z)$ and $Tab(z + \delta^{r1})$ being regular tableau or derivative tableau, we have to look at different coefficients as shows the following table:

Type	$Tab(z)$	$Tab(z + \delta^{r1})$	Coefficient of $Tab(z + \delta^{r1})$
(a)	$T(\bar{v} + z)$	$T(\bar{v} + z + \delta^{r1})$	$\mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})e_{r,r+1}(v + z))$
(b)	$\mathcal{DT}(\bar{v} + z)$	$\mathcal{DT}(\bar{v} + z + \delta^{r1})$	$e_{r,r+1}(\bar{v} + z)$
(c)	$T(\bar{v} + z)$	$\mathcal{DT}(\bar{v} + z + \delta^{r1})$	$((v_{ki} - v_{kj})e_{r,r+1}(v + z))(\bar{v})$
(d)	$\mathcal{DT}(\bar{v} + z)$	$T(\bar{v} + z + \delta^{r1})$	$\mathcal{D}^{\bar{v}}(e_{r,r+1}(v + z))$

- (i) Consider a tableaux $Tab(z)$ such that $(\bar{v} + z)_{ki} = (\bar{v} + z)_{k-1,t} = x$ and $(\bar{v} + z)_{kj} = x + a$, with $a \in \mathbb{Z}$. We will represent this part of tableau $Tab(z)$ by (the row where appear two variable equal to x is the k -th row of the tableau $Tab(z)$).

$$Tab(z) = \begin{pmatrix} x & x+a \\ & x \end{pmatrix}$$

This tableau can be regular (if $a \geq 0$) or derivative (if $a < 0$). We will analyse this two cases. When $E_{k-1,k}$ acts in this tableau we obtain the following tableau:

$$Tab(z + \delta^{k-1,t}) = \begin{pmatrix} x & x+a \\ & x+1 \end{pmatrix}$$

Note that

$$|\Omega^+(Tab(z + \delta^{k-1,t}))| = \begin{cases} |\Omega^+(Tab(z))| - 2, & \text{if } a = 0 \\ |\Omega^+(Tab(z))| - 1, & \text{otherwise.} \end{cases}$$

For $a \geq 0$ the tableaux $Tab(z)$ is regular. In this case we will analyse the coefficients of types (a) and (c). Recall that,

$$(16) \quad e_{k-1,k}(v+z) = -\frac{\prod_{t=1}^k ((v+z)_{k-1,1} - (v+z)_{kt})}{\prod_{t \neq 1}^{k-1} ((v+z)_{k-1,1} - (v+z)_{k-1,t})}$$

In this case, $e_{k-1,k}(v+z)$ is smooth function (because the singularity is in k -th row and in the denominator appear differences between elements of the $(k-1)$ -th row). Thus, by Lemma (4.3) follows that:

- $\mathcal{D}^{\bar{v}}((v_{ri} - v_{rj})e_{k-1,k}(v+z)) = e_{k-1,k}(\bar{v}+z)$, and, as in this case we have the relation $(\bar{v}+z)_{kj} = (\bar{v}+z)_{k-1,t}$, therefore $e_{k-1,k}(\bar{v}+z) = 0$.
- $((v_{ki} - v_{kj})e_{k-1,k}(v+z))(\bar{v}) = 0$.

If $a < 0$, the tableau $Tab(z)$ is derivative tableau. In this case, we need analyze the coefficients of types (b) and (d). Now the coefficients of tableau $Tab(z + \delta^{k-1,t})$ are $e_{k-1,k}(\bar{v}+z)$ and $\mathcal{D}^{\bar{v}}(e_{k-1,k}(v+z))$. In this case, we have that

- $e_{k-1,k}(\bar{v}+z) = 0$, because in the numerator of this rational function appear a difference $(\bar{v}+z)_{kj} - (\bar{v}+z)_{k-1,t}$ that is equal to zero, in this case.
- For the coefficient $\mathcal{D}^{\bar{v}}(e_{k-1,k}(\bar{v}+z))$, we have the following:

$$\begin{aligned} \mathcal{D}^{\bar{v}}(e_{k-1,k}(\bar{v}+z)) &= \mathcal{D}^{\bar{v}}(-(v_{k-1,t} - (v_{ki} + a))(v_{k-1,t} - v_{ki})\varphi(v)) \\ &= -\frac{1}{2}\varphi(\bar{v})a \end{aligned}$$

where $\varphi(v)$ is a rational function of the entries of the vector v that not depends of the entries v_{ki} end v_{kj} , moreover $\varphi(\bar{v}) \neq 0$. As $a \neq 0$, in this case we have $\mathcal{D}^{\bar{v}}(e_{k-1,k}(\bar{v}+z)) \neq 0$.

- (ii) Now we will consider a tableau $Tab(z)$ such that $k \neq n-1$, $(\bar{v}+z)_{k+1,t} = (\bar{v}+z)_{ki} = x$ and $(\bar{v}+z)_{kj} = x+a$, with $a \in \mathbb{Z}$. A representation of part this tableau is

$$Tab(z) = \begin{pmatrix} & x & \\ x & & x+a \end{pmatrix}$$

This tableau should be regular or derivative depending on the value of a . We will analyze this two cases. When $E_{k,k+1}$ acts in this tableau we obtain the tableau

$$Tab(z + \delta^{k,t}) = \begin{pmatrix} & x & \\ x+1 & & x+a \end{pmatrix}$$

In this case, we have that $|\Omega^+Tab(z + \delta^{k,t})| = |\Omega^+Tab(z)| - 1$. Initially we will assume that this tableau is regular tableau ($a \geq 0$) and we will analyze the coefficients of type (a) and (c). For this, recall that:

$$(17) \quad e_{k,k+1}(v+z) = -\frac{\prod_{t=1}^{k+1} ((v+z)_{k,1} - (v+z)_{k+1,t})}{\prod_{t \neq 1}^k ((v+z)_{k,1} - (v+z)_{kt})}.$$

If $a > 0$, in this case that $e_{k,k+1}(v+z)$ is a smooth function, then by Lemma 4.3 follows that

- $\mathcal{D}^{\bar{v}}((v_{ri} - v_{rj})e_{k,k+1}(v+z)) = e_{k,k+1}(\bar{v}+z)$, and, as we have the relation $(\bar{v}+z)_{k+1,t} = (\bar{v}+z)_{k-1,1}$ then, $e_{k,k+1}(\bar{v}+z) = 0$.
- $((v_{ki} - v_{kj})e_{k,k+1}(v+z))(\bar{v}) = 0$.

On the other hand, if $a = 0$, we have

$$\begin{aligned} ((v_{k1} - v_{kj})e_{k-1,k}(v+z))(\bar{v}) &= \left(\frac{\prod_{t=1}^k ((v+z)_{k-1,1} - (v+z)_{kt})}{\prod_{t \neq 1,j}^{k-1} ((v+z)_{k-1,1} - (v+z)_{k-1,t})} \right) (\bar{v}) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathcal{D}^{\bar{v}}((v_{ki} - v_{kj})e_{k-1,k}(v+z)) &= \mathcal{D}^{\bar{v}} \left(\frac{\prod_{t=1}^k ((v+z)_{k-1,1} - (v+z)_{kt})}{\prod_{t \neq 1,j}^{k-1} ((v+z)_{k-1,1} - (v+z)_{k-1,t})} \right) \\ &= 0 \end{aligned}$$

Finally, if $a < 0$ the tableau $Tab(\bar{v} + z)$ is a derivative tableau. In this case we will analyze the coefficients of type (b) and (d). In this case, using the formula (17), follows that:

- $e_{k-1,k}(\bar{v} + z) = 0$, because in the numerator of this rational function appear the difference $(\bar{v} + z)_{k1} - (\bar{v} + z)_{k+1,1}$, that is equal to zero in this case.
- More one time using the formula (17), we have that $\mathcal{D}^{\bar{v}}(e_{k-1,k}(\bar{v} + z)) \neq 0$.

Continuing this analysis, case by case, we can identify all tableaux $Tab(w) \in U \cdot Tab(z)$ such that the correspondent coefficient is no zero is a tableaux $Tab(w)$ such that $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$, that was described in begin the poof of Lemma 7.4 (cases (I)-(V)). Moreover, for all tableaux $Tab(w) \in V(T(\bar{V}))$ such that $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(z))| - 2$ have respective coefficients equal to zero.

□

The following lemma shows that, for each of the cases described in Lemma 7.4 where $Tab(z) \prec Tab(w)$ and $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$, (cases (I)-(V)), the action of the basis elements of $\mathfrak{gl}(n)$ on $Tab(w)$ will generate tableaux $Tab(w')$ such that $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(w'))|$ (i.e. $Tab(w) \prec_g Tab(w')$ for some $g \in \mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ implies $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(w'))|$).

Lemma 7.5. *Let $Tab(w)$ be a tableau such that $Tab(z) \prec Tab(w)$ and $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$ for some $Tab(z)$. If $Tab(w) \prec_g Tab(w')$ for some $g \in \mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$, then $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(w'))|$.*

Proof. By Lemma 7.4 the set of all possible tableaux $Tab(w)$ satisfying the condition is given by:

$$(1) \begin{pmatrix} x & & x+a \\ & x+1 & \end{pmatrix}; \begin{pmatrix} & x & \\ x+a & & x+1 \end{pmatrix}; \begin{pmatrix} x-1 & & x+a \\ & x & \end{pmatrix}$$

with $a \in \mathbb{Z}_{<0}$.

$$(2) \begin{pmatrix} & x & \\ x & & x+1 \end{pmatrix}; \begin{pmatrix} x-1 & & x \\ & x & \end{pmatrix}.$$

Now, for each $Tab(w)$ as before we will construct subsets $W(Tab(w))$ and $W^*(Tab(w))$, of $\mathcal{B}(T(\bar{v}))$ such that:

- (i) $Tab(w) \in W(Tab(w)) \cup W^*(Tab(w))$ and $Tab(w') \in W(Tab(w)) \cup W^*(Tab(w))$ implies $|\Omega^+(Tab(w'))| \geq |\Omega^+(Tab(w))|$.
- (ii) $W(Tab(w)) \cup W^*(Tab(w))$ is $\mathfrak{gl}(n)$ -invariant.

The construction before will be enough to finish the proof. In fact, if $Tab(w) \prec_g Tab(w')$ with $g \in \mathfrak{gl}(n)$, then $Tab(w') \in W(Tab(w)) \cup W^*(Tab(w))$ (because of condition (ii)) and, then $|\Omega^+(Tab(w'))| \geq |\Omega^+(Tab(w))|$ (because of condition (i)).

For instance, consider $Tab(w)$ to be of the form $\begin{pmatrix} x & & x+a \\ & x+1 & \end{pmatrix}$. Define the following subsets of $\mathcal{B}(T(\bar{v}))$:

$$W(Tab(w)) = \left\{ \begin{pmatrix} x+b & & x+a+c \\ & x+1+d & \end{pmatrix} \mid b-1-d < 0 \text{ and } b+a-1-d < 0 \right\}$$

$$W^*(Tab(w)) = \left\{ \begin{pmatrix} x+b & & x+a+c \\ & x+1+d & \end{pmatrix} \mid b-1-d \geq 0 \text{ or } b+a-1-d \geq 0 \right\}.$$

Note that, the tableaux in $W(Tab(w))$ or $W^*(Tab(w))$ can be regular or derivative. It is easy to check that $W(Tab(w)) \cup W^*(Tab(w))$ is $\mathfrak{gl}(n)$ -invariant. Also, $Tab(w') \in W(Tab(w)) \cup W^*(Tab(w))$ satisfies $|\Omega^+(Tab(w'))| \geq |\Omega^+(Tab(w))|$. In fact, if $Tab(w') \in W(Tab(w))$ follows that $|\Omega^+(Tab(w'))| = |\Omega^+(Tab(w))|$ and, by on the other hand, if $Tab(w') \in W^*(Tab(w))$ follows that $|\Omega^+(Tab(w'))| > |\Omega^+(Tab(w))|$.

The construction of $W(Tab(w))$ and $W^*(Tab(w))$ for the other cases is analogous. □

Proof of Lemma 7.3. We will show that if $Tab(w) \in E_1 \cdots E_t \cdot Tab(z)$, then

$$(18) \quad |\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1,$$

and by linearity the result follows for every element of $u \in U(\mathfrak{gl}(n))$.

Proof. Let $Tab(z) \in V(T(\bar{v}))$ and $E_1 \cdots E_t \in U$. We will use induction over t . Indeed, if $t = 1$, this result is true by Lemma 7.4. Now, suppose that the lemma is true for every s such that $1 \leq s \leq t-1$. We will proof that the result is true for t . Indeed, as

$$E_1 E_2 \cdots E_{t-1} E_t \cdot Tab(z) = E_1 E_2 \cdots E_{t-1} (E_t \cdot Tab(z))$$

we have to consider two cases:

Case 1. If $Tab(z)$ is not a tableau described in Lemma 7.4 (cases (I)-(V)), follows that $E_t \cdot Tab(z) = \sum_i \alpha_i Tab(w_i)$, where $|\Omega^+(Tab(w_i))| \geq |\Omega^+(Tab(z))|$. In this case, by induction hypothesis, the action of $E_1 E_2 \cdots E_{t-1}$ over $(E_t \cdot Tab(w))$ we will obtain a linear combination of tableaux $Tab(w'_i)$ such that $|\Omega^+(Tab(w'_i))| \geq |\Omega^+(Tab(z))| - 1$ which proves the desired result in this case.

Case 2. Now we will assume that $Tab(z)$ is a tableau described in Lemma 7.4 (cases (I)-(V)). In this case follows that

$$E_t \cdot Tab(z) = \sum_i \alpha_i Tab(w_i) + \sum_i \beta_i Tab(w'_i)$$

where $|\Omega^+(Tab(w_i))| \geq |\Omega^+(Tab(z))|$ and $|\Omega^+(Tab(w'_i))| = |\Omega^+(Tab(z))| - 1$. Thus, by induction hypothesis, follows that when $E_1 E_2 \cdots E_{t-1}$ acts in tableaux $Tab(w_i)$ we obtain a linear combination of new tableaux $Tab(w''_i)$ such that $|\Omega^+(Tab(w''_i))| \geq |\Omega^+(Tab(z))| - 1$. By other hand, when $E_1 \cdots E_{t-1}$ acts in tableaux $Tab(w'_i)$, we will get a linear combination of tableaux belong the set $W(Tab(w)) \cup W^*(Tab(w))$ (that

was described in Lemma 7.5). Thus, in this case we have that in decomposition $E_1 \cdots E_{t-1} \cdot Tab(w'_i)$ will appear tableaux $Tab(w_i^*)$ such that

$$|\Omega^+(Tab(w_i^*))| = |\Omega^+(Tab(z))| - 1$$

which prove the result in this second case. \square

Corollary 7.6. *If $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ is such that $|\Omega^+(Tab(z))| \geq 1$ then, $U \cdot Tab(z)$ is a proper submodule of $V(T(\bar{v}))$.*

Proof. Indeed, if the module $U \cdot Tab(z)$ is not a proper submodule of $V(T(\bar{v}))$, follows that for every $Tab(w) \in V(T(\bar{v}))$ we have $Tab(z) \prec Tab(w)$. But by Lemma(7.3), follows that $|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1$. We have two cases to consider:

Case 1. If $|\Omega^+(Tab(z))| \geq 2$, follows that

$$|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1 \geq 2 - 1 = 1$$

but, this is a contradiction, we always have $T(w) \in V(T(\bar{v}))$ such that $|\Omega^+(Tab(w))| = 0$.

Case 2. If $|\Omega^+(Tab(z))| = 1$, follows that the unique integer difference $\bar{v}_{rs} - \bar{v}_{r-1,t}$ is not close to critical line, because if this integer difference is close to critical line we would $|\Omega^+(Tab(z))| \geq 2$. But away from the critical line we have the generic case, where we have the following inequality

$$|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| = 1$$

As in the last case this is a contradiction.

Hence, if $|\Omega^+(Tab(z))| \geq 1$ the module $U \cdot Tab(z)$ could not be the full module $V(T(\bar{v}))$, thus in this case, $U \cdot Tab(z)$ is a proper submodule. \square

Theorem 7.7. *If $V(T(\bar{v}))$ is irreducible then, $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$ for any $1 \leq s \leq r \leq n$ and $1 \leq t \leq r - 1$.*

Proof. Suppose that $\bar{v}_{rs} - \bar{v}_{r-1,t} \in \mathbb{Z}$ for some $1 \leq t < r \leq n, 1 \leq s \leq r$, choose $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ such that $|\Omega^+(Tab(w))|$ is maximal ($|\Omega^+(Tab(w))| \geq 1$) and consider the submodule N of $V(T(\bar{v}))$ generate by $Tab(w)$. Thus by 7.6 follows that the submodule N is proper. \square

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